

Geometry I, 2006 : Coursework 5 Solutions

1. (i) [1 mark] The line ℓ through the points $(1, 2, 3)$ and $(4, 5, 6)$ is the line through the point $(1, 2, 3)$ with direction that of the vector $\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$. Thus parametric equations for ℓ are

$$\begin{cases} x &= 1 &+& 3\lambda \\ y &= 2 &+& 3\lambda \\ z &= 3 &+& 3\lambda \end{cases}.$$

We find the intersection of the given plane with ℓ by solving

$$2(1 + 3\lambda) - 3(2 + 3\lambda) + 6(3 + 3\lambda) = 4$$

for λ . This gives $15\lambda = -10$, or $\lambda = -2/3$.

[1 mark] Substituting this λ into the parametric equations for ℓ , we get the intersection to be

$$\{(-1, 0, 1)\}$$

(consisting of the single point $(-1, 0, 1)$).

(ii) [1 mark] The parametric equations for m are

$$\begin{cases} x &= -2 &+& 4\mu \\ y &= 1 &+& 2\mu \\ z &= 1 &+& 3\mu \end{cases}.$$

To determine the intersection of ℓ with m we solve

$$\begin{cases} 1 &+& 3\lambda &= &-2 &+& 4\mu \\ 2 &+& 3\lambda &= &1 &+& 2\mu \\ 3 &+& 3\lambda &= &1 &+& 3\mu \end{cases}$$

for λ and μ . We obtain the following system of linear equations:

$$\begin{cases} 3\lambda &-& 4\mu &= &-3 \\ 3\lambda &-& 2\mu &= &-1 \\ 3\lambda &-& 3\mu &= &-2 \end{cases}.$$

Applying Gaussian elimination, we find that this system is equivalent to

$$\begin{cases} 3\lambda &-& 4\mu &= &-3 \\ && 2\mu &= &2 \end{cases}.$$

Thus $\mu = 1$, $\lambda = (-3 + 4)/3 = 1/3$ is the only solution.

[1 mark] Substituting $\mu = 1$ into the parametric equations for m , we get the intersection to be

$$\{(2, 3, 4)\}$$

(consisting of the single point $(2, 3, 4)$).

2. [3 marks] [The proof is obtained by careful modification of the proof that $(\alpha \underline{u}) \times \underline{v} = \alpha(\underline{u} \times \underline{v})$.]

If $\underline{u} = \underline{0}$ or $\underline{v} = \underline{0}$ or $\underline{u}, \underline{v}$ parallel or $\alpha = 0$, then $\underline{u} \times (\alpha \underline{v}) = \underline{0} = \alpha(\underline{u} \times \underline{v})$.

Otherwise, let θ be the angle between $\underline{u}$ and $\underline{v}$, and let $\underline{w} = \underline{u} \times \underline{v}$. The angle between $\underline{u}$ and $\alpha \underline{v}$ is θ if $\alpha > 0$ and $\pi - \theta$ if $\alpha < 0$, and we have

$$|\alpha \underline{w}| = |\alpha| |\underline{w}| = |\alpha| |\underline{u}| |\underline{v}| \sin \theta = |\underline{u}| |\alpha \underline{v}| \sin \theta = |\underline{u}| |\alpha \underline{v}| \sin(\pi - \theta).$$

Moreover, $\alpha \underline{w}$ is orthogonal to $\underline{u}$ and $\alpha \underline{v}$ (since $\underline{w}$ is orthogonal to $\underline{u}$ and $\underline{v}$), and $\underline{u}, \alpha \underline{v}, \alpha \underline{w}$ is a right-handed triple (whether $\alpha > 0$ or $\alpha < 0$). Thus $\underline{u} \times (\alpha \underline{v}) = \alpha \underline{w} = \alpha(\underline{u} \times \underline{v})$.

3. [1 mark each; some calculation required]

(i) $(\underline{i} \times \underline{j}) \times \underline{j} = \underline{0} \times \underline{j} = \underline{0}$

[Note: the answer is the zero vector $\underline{0}$, not the number 0.]

(ii) $\underline{i} \times (\underline{i} \times \underline{j}) = \underline{i} \times \underline{k} = -\underline{j}$

(iii) $(\underline{k} \times \underline{j}) \times (\underline{j} \times \underline{i}) = (-\underline{i}) \times (-\underline{k}) = \underline{i} \times \underline{k} = -\underline{j}$

(iv) $(\underline{i} \times \underline{j}) \cdot \underline{k} = \underline{k} \cdot \underline{k} = |\underline{k}|^2 = 1^2 = 1$

[Note: The answer is the number 1, not a vector.]

4. [1 mark for reasonably correct calculations; 1 mark for correct final answer]

The line has vector equation $\underline{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$.

Let $A = (2, -1, 3)$, $P = (-1, 1, 3)$, $\underline{u} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$, and let $\underline{v}$ be represented by $\overrightarrow{AP}$. Then

$$\underline{v} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}, \quad \underline{u} \times \underline{v} = \begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} \underline{i} - \begin{vmatrix} -1 & -3 \\ 4 & 0 \end{vmatrix} \underline{j} + \begin{vmatrix} -1 & -3 \\ 2 & 2 \end{vmatrix} \underline{k} = -8\underline{i} - 12\underline{j} + 4\underline{k},$$

and the distance from P to the line is $|\underline{u} \times \underline{v}| / |\underline{u}| = \sqrt{64 + 144 + 16} / \sqrt{1 + 4 + 16} = \sqrt{224} / \sqrt{21} = 4\sqrt{2} / \sqrt{3}$.

5. (i) [1 mark for reasonably correct calculations; 1 mark for correct final answer]

Let $\underline{a} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$, $\underline{u} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$, $\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Then $\underline{u} \times \underline{v} = \begin{vmatrix} 3 & 2 \\ -4 & 3 \end{vmatrix} \underline{i} - \begin{vmatrix} 1 & 1 \\ -4 & 3 \end{vmatrix} \underline{j} + \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} \underline{k} = 17\underline{i} - 7\underline{j} - \underline{k}$, and the distance between the lines is $|(\underline{b} - \underline{a}) \cdot (\underline{u} \times \underline{v})|/|\underline{u} \times \underline{v}| = \left| \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 17 \\ -7 \\ -1 \end{pmatrix} \right| / \sqrt{289 + 49 + 1} = 33/\sqrt{339}$.

(ii) [1 mark for reasonably correct calculations; 1 mark for correct final answer]

These lines are parallel (since their direction vectors are parallel), and so the distance between them is the distance between one, say $\underline{r} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$, and a point P , say $(2, 3, 6)$, on the other.

Let $A = (-1, 1, 2)$, $P = (2, 3, 6)$, $\underline{u} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$, and let $\underline{v}$ be represented by $\overrightarrow{AP}$. Then $\underline{v} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $\underline{u} \times \underline{v} = \begin{vmatrix} 3 & 2 \\ -4 & 4 \end{vmatrix} \underline{i} - \begin{vmatrix} 1 & 3 \\ -4 & 4 \end{vmatrix} \underline{j} + \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} \underline{k} = 20\underline{i} - 16\underline{j} - 7\underline{k}$, and the distance from P to the line is $|\underline{u} \times \underline{v}|/|\underline{u}| = \sqrt{400 + 256 + 49}/\sqrt{1 + 9 + 16} = \sqrt{705}/\sqrt{26}$.

6.

[1 mark] Let $P = (1, -2, 3)$, $Q = (-5, 0, 1)$, $R = (2, 1, -1)$, let $\underline{u}$ be represented by $\overrightarrow{PQ}$ and let $\underline{v}$ be represented by $\overrightarrow{PR}$. Then $\underline{u} = \begin{pmatrix} -6 \\ 2 \\ -2 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$.

We require a Cartesian equation of the plane through $A = (1, 2, 1)$ and parallel to $\underline{u}$ and $\underline{v}$.

[1 mark] A vector equation for this plane is $\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$, where $\underline{n} = \underline{u} \times \underline{v}$ and $\underline{a}$ is the position vector of A .

We have $\underline{n} = \underline{u} \times \underline{v} = -2\underline{i} - 26\underline{j} - 20\underline{k}$, and so $\underline{a} \cdot \underline{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -26 \\ -20 \end{pmatrix} = -74$.

[1 mark] Thus, a vector equation for the plane is $\underline{r} \cdot \begin{pmatrix} -2 \\ -26 \\ -20 \end{pmatrix} = -74$, and a Cartesian equation is $-2x - 26y - 20z = -74$, equivalently $x + 13y + 10z = 37$.